2 Essay on Spectra

Ever since the first definition of *homotopy groups of topological spaces* by Hurewitz and Čech in the 1930s, mathematicians have been diligently computing the homotopy groups of spheres. The challenge presents itself in multiple aspects. However, one of the biggest challenge among all would be the difference between homotopy groups and *homology groups of spaces*, where the latter is a much computable invariant on topological spaces through practices.

Accepting the fundamental difference between homotopy and homology theories on spaces, a large group of topologists focused *stable homotopy groups* thanks to Freudenthal's suspension Theorem. Then, to find a theory that can nicely situate stable homotopy groups, *spectra* and corresponding categories are built. Despite the presence of multiple models, spectra that can be viewed as a relaxation of topological space in general. Although building the framework requires very technical treatment, stable homotopy theory (working with spectra) yields fruitful properties on stable homotopy groups, provides rigorous analogy between algebra and topology, and connects homotopy and homology theories.

The first, possibly the biggest, advantage of working with spectra is a property called *stability.* Recall that in homology theory in spaces, *suspension* is an operation that hops the homological degree by one. Indeed, similar to suspension, operations like *mapping cone, path space* and so many other topological construction does not interact well with homotopy groups. However, stability theorems on spectra ensure that they behave nicely with stable homotopy groups. Crucially, taking suspension on spectra is an operation that maintains the stable homotopy theory without any extra condition. Furthermore, taking mapping cone of the path *fibration* and taking path space of mapping *cofibration* are also equivalent in the eyes of stable homotopy groups. All of the above properties fail easily in the classical framework of topological spaces.

Secondly, more modern models of spectra like *symmetric spectra* and *orthogonal spectra* gives *closed Cartesian monoidal structure* that enables spectra to multiply with each other. This allows considering familiar algebraic structure like *monoids, modules*, and *algebras*, which are called *ring spectra, module spectra,* and *algebra spectra* respectively in stable homotopy theory. Other than the naming, there is a stronger connection between traditional algebraic categories and corresponding stable homotopy categories. For example, there is a functor called taking *Eilenberg-MacLane spectra* that associate each commutative ring R to a commutative ring spectra HR. Amazingly, the homotopy theory of module spectra over HR is equivalent to the homotopy theory of chain complexes over R [2]; and the homotopy theory of algebra spectra over HR is equivalent to the homotopy theory of differential graded algebras over R [3]. Those results are the foundations of "higher algebra," which is the study of ring, module, and algebra spectra up to stable homotopy equivalence. Although they give the same homotopy theory, the spectra side contains richer information than the classical algebraic categories.

Last but not least, stable homotopy theory is a convenient language for connecting stable homotopy theory and generalized homology theory. Through a categorical technical called *Bousfield localizations*, spectra up to homology equivalence becomes the framework for the connection. Under mild assumptions (sphere spectrum satisfies), there is a spectral sequence called *the Adam spectral sequence* that has the homology theory converges to stable homotopy groups [4]. Worth noting that this technology is still used actively in research for computing most of the known stable homotopy groups of spheres.

In conclusion, spectra stable homotopy theory provides working algebraic topologists with a modern program for studying topological phenomena up to homotopy.

References

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